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On computational search for optimistic solutions in bilevel problems

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Abstract The linear-linear and quadratic-linear bilevel programming problems are considered. Their optimistic statement is reduced to a nonconvex mathematical programming problem with the bilinear structure. Approximate algorithms of local and global search in the obtained problems are proposed. The results of computational solving randomly generated test problems are given and analyzed.

Keywords Bilevel programming · Optimistic solution · Nonconvex optimization problems · Local search · Global search · Computational simulation

1 Introduction

During the recent decades, problems with hierarchical structure seem to be the most attractive field for many experts [1–3]. In particular, problems of bilevel programming represent extreme problems, which—side by side with ordinary constraints such as equalities and inequalities—include a constraint described as an optimization subproblem [2,3]. Problems of this kind are rather complex from the viewpoint of investigation since there appears the need to take simultaneous account of the goals of different levels of hierarchy.

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In course of investigation of bilevel programming problems the difficulty arises already at the stage of defining the concept of solution. The optimistic and pessimistic (guaranteed) concepts of solution are known to be the most popular [1–3].

Note, bilevel programming densely contacts with such domains of MPCC (mathematical programming with complementarity constraints) and MPEC (mathematical programming with eqiulibrium constraints) [4].

During the three decades of intensive investigation of bilevel programming problems there were many methods of finding the optimistic solutions proposed by different authors (see the surveys [5,6]). Nevertheless, as far as we can conclude on the basis of available literature, there are few published results containing numerical solutions of even test bilevel high-dimension problems (e.g. problems with the dimension up to 200). Most frequently authors restrict their consideration with illustrative examples with the dimension up to 10 (see [7,8]) and only in [9–11] one can find some results of solving nonlinear bilevel problems with the dimension up to 30. So, development of new numerical methods even for the simplest classes of nonlinear bilevel problems, while implying verification of their efficiency by numerical testing, is one of the most essential problems of operations research.

This paper is devoted to elaboration of new techniques of finding optimistic solutions of bilevel problems, where the upper level goal function is either linear or convex quadratic, and the lower level goal function is linear. Such a bilevel problem may be reduced to one or several single-level problems via, for instance, the KKT-rule (see, for example, [2]). Furthermore, it turns out that the reduced problems turn out to be nonconvex, and, therefore, such problems may have a large number of local solutions, which are far—even from the viewpoint of the goal function's value—from a global one. Therefore, in bilevel problems the nonconvexity is not immediately visible, implicit and hidden. Direct application of standard convex optimization methods [12,13] turns out to be inefficient from the view point of global search. So, there appears the need to construct new global search methods [14–16], allowing to escape from a stationary point.

For the purpose of solving the problems formulated above, we intend to construct the algorithms based on the Global Search Theory developed in [16–24], and oriented to obtaining the global solution.

The structure of this paper is as follows. Section 2 describe the statement of the bilevel problem, and its reduction to nonconvex problems of nonlinear programming is discussed. Next, in the Sects. 3 and 4, we propose methods of local and global search for these problems, respectively. Results of numerical solving test bilevel problems are described in Sect. 5.

2 Problem statement and its reduction

Consider the following bilevel programming problem in its optimistic statement:

$$\left.\begin{array}{c} F(x,y) \triangleq \frac{1}{2}\langle x, Cx\rangle + \langle c, x\rangle + \frac{1}{2}\langle y, C_1 y\rangle + \langle c_1, y\rangle \downarrow \min\limits_{x,y}, \\ (x,y) \in X \triangleq \{(x,y) \in I\!R^m \times I\!R^n \mid Ax + By \le a, \quad x \ge 0\}, \\ y \in Y_*(x) \triangleq \operatorname{Argmin}\limits_{y}\{\langle d, y\rangle \mid \quad y \in Y(x)\}, \\ Y(x) \triangleq \{y \in I\!R^n \mid A_1 x + B_1 y \le b, \quad y \ge 0\}, \end{array}\right\} \qquad (\mathcal{BP})$$

where $A \in I\!R^{p\times m}$, $B \in I\!R^{p\times n}$, $A_1 \in I\!R^{q\times m}$, $B_1 \in I\!R^{q\times n}$, $C \in I\!R^{m\times m}$, $C = C^T > 0$, $C_1 \in I\!R^{n\times n}$, $C_1 = C_1^T > 0$, $c \in I\!R^m$, $c_1, d \in I\!R^n$, $a \in I\!R^p$, $b \in I\!R^q$. When $C \equiv 0$ and $C_1 \equiv 0$ the problem represents one of simple cases, i.e. a linear-linear bilevel problem.

Suppose, function $F(x, y)$ is lower bounded on set X, and function $\langle d, y\rangle$ is lower bounded on set $Y(x)$ for any $x \in Pr(X) \triangleq \{x \in I\!R^m \mid \exists y : (x, y) \in X\}$ so that $\inf_x \inf_y \{\langle d, y\rangle \mid y \in Y(x),\ x \in Pr(X)\} > -\infty$.

In order to execute the reduction of problem $(\mathcal{BP})$ to a single-level problem we use a conventional technique of replacement of the lower-level problem in $(\mathcal{BP})$ with the equivalent Karush-Kuhn-Tucker (KKT) conditions [2,3]. As a result, we obtain the following problem:

$$\left.\begin{array}{c} F(x, y) \downarrow \min\limits_{x,y,v}, \\ (x, y, v) \in D \triangleq \{(x, y, v) \mid Ax + By \le a,\ A_1x + B_1y \le b, \\ d + vB_1 \ge 0,\ (x, y, v) \ge 0\}, \\ h(x, y, v) \triangleq \langle d, y\rangle - \langle A_1x - b, v\rangle = 0, \end{array}\right\} \qquad (\mathcal{P})$$

where $v \in I\!R^q$ are Lagrange multipliers. Note, that the dimension of the problem $(\mathcal{P})$ is now $m + n + q$, instead of $m + n$ in $(\mathcal{BP})$

Theorem 1 [2,3] *The pair (x^*, y^*) is a global optimistic solution to the bilevel problem $(\mathcal{BP})$ if and only if there exists a vector $v^* \in I\!R^q$, $v^* \ge 0$ such that the triplet (3-tuple) (x^*, y^*, v^*) is a global solution of problem $(\mathcal{P})$.*

Note, at the expense of presence of the bilinear equality constraint, the feasible set of problem $(\mathcal{P})$ turns out to be nonconvex. Therefore, the search for an optimistic solutions to the initial bilevel programming problem turns out to be equivalent to solving the nonconvex optimization problem.

It can be readily demonstrated with the use of the duality theory [13] that

$$h(x, y, v) \triangleq \langle d, y\rangle - \langle A_1x - b, v\rangle \ge 0 \quad \forall (x, y, v) \in D. \qquad (1)$$

The naturally structure of nonconvexity presented in problem $(\mathcal{P})$ and generated by the bilinear equality constraint naturally imply the following two approaches.

Firstly, problem $(\mathcal{P})$ may be considered directly, as a problem with d.c. inequality constraint [16,18,20,22] (i.e. with a constraint represented as a difference of two convex functions). Indeed, on account of inequality (1), the bilinear equality constraint may, for example, be replaced with the inequality

$$\langle d, y\rangle - \langle A_1x - b, v\rangle \le 0, \qquad (2)$$

which holds for the vectors $(x, y, v) \in D$ (see $(\mathcal{P})$) only when $\langle d, y\rangle = \langle A_1x - b, v\rangle$. Therefore, in order to find a solution for the bilevel problem $(\mathcal{BP})$, we have to solve the following problem:

$$F(x, y) \downarrow \min_{x,y,v}, \quad \in D, \quad h(x, y, v) \le 0. \qquad (\mathcal{DCC})$$

As far as this class of problems is concerned, we have developed a Global Search Strategy, which is based on Global Optimality Conditions and can be found in [18,20], as well as special Local Search methods proposed in [22], which have already proved their efficiency.

Secondly, problem $(\mathcal{P})$ may be reduced to a family of problems with a convex feasible set and the nonconvex d.c. goal function:

$$\Phi(x, y, v) \triangleq F(x, y) + \mu h(x, y, v) \downarrow \min_{x,y,v}, \quad (x, y, v) \in D, \qquad (\mathcal{DC}(\mu))$$

where $\mu > 0$ is a penalty parameter. It can be shown on account the above mentioned assumptions on problem $(\mathcal{BP})$ and inequality (1) that the goal function of problem $(\mathcal{DC}(\mu))$ is lower bounded on the set D.

Let $(x(\mu), y(\mu), v(\mu))$ be a solution of problem ($\mathcal{DC}(\mu)$) corresponding to some fixed parameter $\mu > 0$. Introduce the denotation $h[\mu] \triangleq h(x(\mu), y(\mu), v(\mu))$. The following assertion sets the relationships between problem ($\mathcal{DC}(\mu)$) and problem ($\mathcal{P}$).

Proposition 1 [13].

i) *Let for some $\mu = \hat{\mu}$ the equality $h(x(\hat{\mu}), y(\hat{\mu}), v(\hat{\mu})) = 0$ be satisfied for the solution $(x(\hat{\mu}), y(\hat{\mu}), v(\hat{\mu}))$ of problem ($\mathcal{DC}(\mu)$). Hence the triplet $(x(\hat{\mu}), y(\hat{\mu}), v(\hat{\mu}))$ turns out to be a solution of problem ($\mathcal{P}$).*

ii) *For any value of parameter $\mu > \hat{\mu}$ the function $h[\mu]$ turns zero, i.e. $h(x(\mu), y(\mu), v(\mu)) = 0$, and $(x(\mu), y(\mu), v(\mu))$ turns out to be a solution of problem ($\mathcal{P}$).*

It can readily be proved that $h[\mu] \to 0$ when $\mu \to +\infty$. The following proposition makes it clear what we obtain if for some μ we have $0 < h[\mu] < \tau$.

Proposition 2 *Let $(x(\mu), y(\mu), v(\mu)) \in D$ be a τ_1-solution of problem ($\mathcal{DC}(\mu)$) and $h(x(\mu), y(\mu), v(\mu)) \leq \tau_2$. Hence*

i) *$y(\mu)$ is a τ_2-solution of the lower-level problem in ($\mathcal{BP}$) with $x = x(\mu)$;*

ii) *$(x(\mu), y(\mu))$ is an approximate τ_1-solution of problem ($\mathcal{BP}$).*

Proof i) The following obvious equalities hold for any point $y \in Y(x(\mu)) \triangleq \{y \geq 0 \mid A_1 x(\mu) + B_1 y \leq b\}$:

$$\begin{aligned}\langle d, y(\mu) - y\rangle &= \langle d, y(\mu)\rangle - \langle d, y\rangle \pm \langle A_1 x(\mu) - b, v(\mu)\rangle\rangle \\ &= h(x(\mu), y(\mu), v(\mu)) - \langle d, y\rangle + \langle A_1 x(\mu) - b, v(\mu)\rangle \\ &= h(x(\mu), y(\mu), v(\mu)) - h(x(\mu), y, v(\mu)). \end{aligned} \tag{3}$$

Having used definitions of D and $Y(x(\mu))$, it can easily be shown that

$$h(x(\mu), y, v(\mu)) \geq 0 \quad \forall y \in Y(x(\mu)). \tag{4}$$

Now let us continue the chain of equalities (3), while taking into account the inequalities $h(x(\mu), y(\mu), v(\mu)) \leq \tau_2$ and (4):

$$h(x(\mu), y(\mu), v(\mu)) - h(x(\mu), y, v(\mu)) \leq \tau_2. \tag{3'}$$

When looking at the beginning and at the end of the chain (3)–(3′), we can conclude that:

$$\langle d, y(\mu)\rangle \leq \langle d, y\rangle + \tau_2 \quad \forall y \in Y(x(\mu)).$$

So, $y(\mu)$ is a τ_2-solution of the lower-level problem in ($\mathcal{BP}$) with $x = x(\mu)$.

ii) Let us to prove the second statement. Having taken (1) into account, we obtain:

$$\begin{aligned} F(x(\mu), y(\mu)) &\leq F(x(\mu), y(\mu)) + \mu h(x(\mu), y(\mu), v(\mu)) \leq \\ &\leq F(x, y) + \mu h(x, y, v) + \tau_1 \quad \forall (x, y, v) \in D. \end{aligned}$$

In particular,

$$F(x(\mu), y(\mu)) \leq F(x, y) + \tau_1 \quad \forall (x, y, v) \in D : h(x, y, v) = 0. \tag{5}$$

Note that the equality $h(x, y, v) = 0$ for $(x, y, v) \in D$ implies that the point y from the triple (x, y, v) is the KKT-point for the convex lower-problem in ($\mathcal{BP}(x)$), and hence y is its solution. Therefore, (5) is equivalent to

$$F(x(\mu), y(\mu)) \leq F(x, y) + \tau_1 \quad \forall (x, y) \in X : y \in Y_*(x).$$

Therefore, the pair $(x(\mu), y(\mu))$ is an approximate τ_1-solution of problem $(\mathcal{BP})$ in the sense that point $y(\mu)$ is a τ_2-solution of the lower-level problem in $(\mathcal{BP})$. □

We propose to apply the Global Search Strategies for d.c. programming problems developed in [16,19] in the capacity of the method for solving the nonconvex problem $(\mathcal{DC}(\mu))$ with a fixed $\mu > 0$.

Local search methods described in the next section turn out to be one of the key elements of the Global Search Strategies in the problems $(\mathcal{DCC})$ and $(\mathcal{DC}(\mu))$.

3 Local search

3.1 The local search method for the problem with a nonconvex (d.c.) equality constraint

For the purpose of local search in problem $(\mathcal{DCC})$ we intend to apply a Special Local Search Method (SLSM) proposed in [22] for such kind of problems. To this end we, first of all, need to obtain an explicit d.c. decomposition of the bilinear function, which assigns the nonconvex constraint. The bilinear constraint in problem $(\mathcal{DCC})$ may be represented in the form of difference between two convex functions, for example, as follows:

$$h(x, y, v) = g(x, y, v) - f(x, v), \tag{6}$$

where $g(x, y, v) = \frac{1}{4}\|A_1 x - v\|^2 + \langle d, y\rangle + \langle b, v\rangle,\ f(x, v) = \frac{1}{4}\|A_1 x + v\|^2$.

The Special Local Search Method from [22] consists of the two procedures. The first one begins from some point $(x, y, v) \in D, h(x, y, v) \leq 0$ and constructs a point $(\bar{x}, \bar{y}, \bar{v}) \in D$ with the properties $h(\bar{x}, \bar{y}, \bar{v}) = 0, F(\bar{x}, \bar{y}) \leq F(x, y)$ (see [22]). So, the obtained point appears to be feasible in problem $(\mathcal{P})$. The second procedure consists in the consecutive (approximate) solving the linearized problems of the form

$$\begin{aligned} (\mathcal{PL}_s(\rho_s)):\quad & g(x, y, v) - \langle \nabla f(x^s, v^s), (x, v)\rangle \downarrow \min_{x,y,v}, \\ & (x, y, v) \in D, \quad F(x, y) \leq \rho_s \triangleq F(x^s, y^s), \end{aligned} \tag{7}$$

which are obviously convex. After finishing the second procedure one obtains a point $(\hat{x}, \hat{y}, \hat{v}) \in D, h(\hat{x}, \hat{y}, \hat{v}) \leq 0$, which represents a solution of the linearized (at the point $(\hat{x}, \hat{y}, \hat{v})$!) problem. Such a point will be called critical (approximately critical) one for problem $(\mathcal{DCC})$. Next, we turn back to procedure 1 again.

Owing to the specific features of the problem $(\mathcal{DCC})$ execution of the second procedure turns out to be excessive, this fact being confirmed by the following result.

Proposition 3 *Any point* $(\hat{x}, \hat{y}, \hat{v})$*, which is feasible in problem* $(\mathcal{P})$*, is a solution of the linearized problem* $(\mathcal{PL}_s(\rho_s))$*-(7), where* $(x^s, v^s) := (\hat{x}, \hat{v}), \rho_s = F(\hat{x}, \hat{y})$*, i.e. this point turns out to be critical for the problem* $(\mathcal{DCC})$.

In other words, SLSM for the problems with a d.c. constraint (for $(\mathcal{DCC})$) degenerates itself into a procedure of finding a feasible point for the problem $(\mathcal{P})$.

3.2 Local search for Problem $(\mathcal{DC}(\mu))$

To the end of a local search for Problem $(\mathcal{DC}(\mu))$ let us apply the idea of consecutive solving partial problems with respect to uncoupled groups of variables (see [17,21,24]). In order to do it, we separate the pair (x, y) and the variable v, so that $D \triangleq Z \times V$, where

$Z = \{(x, y) | A_x + B_y \leq a_1 A_1 x + B_1 y \leq B, x \geq 0, y \geq 0\}$, $V \triangleq \{v \mid d+vB_1 \geq 0, v \geq 0\}$. For a fixed value of variable v problem ($\mathcal{DC}(\mu)$) becomes a convex quadratic optimization problem (or—a linear programming problem if $C \equiv 0$ and $C_1 \equiv 0$), and for a fixed pair (x, y) we obtain a problem of linear programming (LP) with respect to $v \in I\!R^q$. These auxiliary problems can be solved with the help of standard software packages. So, we obtain the following special local search method.

Let $v_0 \in I\!R^q$ be a starting point.

Step 0 Put $s := 0, \quad v^s := v_0$.

Step 1 Apply the technique of quadratic programming and obtain the $\frac{\delta_s}{2}$-solution (x^{s+1}, y^{s+1}) of the problem:

$$\left.\begin{array}{c} \frac{1}{2}\langle x, Cx\rangle + \langle c, x\rangle + \frac{1}{2}\langle y, C_1 y\rangle + \langle c_1, y\rangle + \mu(\langle d, y\rangle - \langle A_1 x, v^s\rangle) \downarrow \min\limits_{(x,y)}, \\ (x, y) \in Z, \end{array}\right\} \qquad (\mathcal{QP}(v^s))$$

Step 2 Obtain the $\frac{\delta_s}{2}$-solution v^{s+1} of the linear program

$$\langle b - A_1 x^{s+1}, v\rangle \downarrow \min_v, \quad v \in V, \qquad (\mathcal{LP}(x^{s+1}))$$

Step 3 Put $s := s + 1$, and go to Step 1.

In order to prove the following theorem of convergence of this method, which is called V-procedure, we henceforth apply

Lemma 1 [12] *Let the numerical sequence* $\{a_s\}$ *be such that*

$$a_{s+1} \leq a_s + \delta_s, \quad \delta_s \geq 0, \quad k = 0, 1, 2, \ldots, \quad \sum_{s=0}^{\infty} \delta_s < +\infty \qquad (8)$$

and bounded from below. Hence there exists a finite limit $\lim\limits_{k \to +\infty} a_k$.

Proof After summation of the first inequalities from (8) we obtain

$$a_r \leq a_s + \sum_{i=s}^{r-1} \delta_i \leq a_s + \sum_{i=s}^{+\infty} \delta_i \quad \forall r > s \geq 0. \qquad (9)$$

Let $\liminf\limits_{s \to +\infty} a_s = \lim\limits_{t \to +\infty} a_{s_t} (s_t < s_{t+1}, t = 0, 1, 2, \ldots)$. Having taken $s = s_t$ in (9), we obtain $a_r \leq a_{s_t} + \sum\limits_{i=s_t}^{+\infty} \delta_i (r > s_t)$. Therefore, $\limsup\limits_{r \to +\infty} a_r \leq a_{s_t} + \sum\limits_{i=s_t}^{+\infty} \delta_i$. Passing to the limit for $t \to +\infty$, we have $\limsup\limits_{r \to +\infty} a_r \leq \lim\limits_{t \to +\infty} a_{s_t} \triangleq \liminf\limits_{r \to +\infty} a_r$. So, the sequence $\{a_s\}$ converges, and finiteness of the limit follows from the fact of its boundedness. □

Theorem 2 i) *If* $\delta_s > 0, s = 0, 1, 2 \ldots, \sum\limits_{s=0}^{\infty} \delta_s < +\infty$, *then the sequence* $\{\Phi_s\}$ *of values of function* $\Phi_s \triangleq \Phi(x^s, y^s, v^s)$, *which is generated by the V-procedure, converges.*

ii) *If* $(x^s, y^s, v^s) \to (\hat{x}, \hat{y}, \hat{v})$, *then the limit point* $(\hat{x}, \hat{y}, \hat{v})$ *satisfies the following inequalities:*

$$\Phi\ (\hat{x}, \hat{y}, \hat{v}) \le \Phi(x, y, \hat{v}) \quad \forall (x, y) \in Z, \tag{10}$$

$$\Phi\ (\hat{x}, \hat{y}, \hat{v}) \le \Phi(\hat{x}, \hat{y}, v) \quad \forall v \in V. \tag{11}$$

Proof i) Denote: $\overline{\Phi}_s \triangleq \Phi(x^{s+1}, y^{s+1}, v^s)$. In accordance with the construction of the method, the following relations ($s \ge 1$):

$$\begin{aligned}\Phi_s \ge \inf_{(x,y)} \{\Phi(x, y, v^s) \mid (x, y) \in Z\} \ge \overline{\Phi}_s - \frac{\delta_s}{2} \ge \\ \ge \inf_v \{\Phi(x^{s+1}, y^{s+1}, v) \mid v \in V\} - \frac{\delta_s}{2} \ge \Phi_{s+1} - \delta_s\end{aligned} \tag{12}$$

are valid. Hence we have

$$\Phi_{s+1} \le \Phi_s + \delta_s, \tag{13}$$

so that the sequence $\{\Phi_s\}$, $s = 0, 1, 2 \ldots$ is almost monotonous nonincreasing and upper bounded. In addition, this sequence is upper bounded because function $\Phi(\cdot)$ is bounded from below over $D \triangleq Z \times V$, and $(x^s, y^s, v^s) \in D$, $s = 1, 2, 3 \ldots$ on account of the construction.

Hence, taking into account the condition imposed on δ_s and using Lemma 1, we conclude that sequence $\{\Phi_s\}$ converges.

ii) According to step 1 of the V-procedure the following inequality is valid:

$$\Phi\left(x^{s+1}, y^{s+1}, v^s\right) - \frac{\delta_s}{2} \le \Phi(x, y, v^s) \quad \forall (x, y) \in Z.$$

When passing to the limit for $s \to +\infty$ ($\delta^s \downarrow 0$) at a fixed point (x, y) in this inequality, and on account of a continuity of function $\Phi(x, y, v)$ we obtain inequality (10). Similarly, according to step 2 of the V-procedure, the following inequality holds:

$$\Phi\left(x^{s+1}, y^{s+1}, v^{s+1}\right) - \frac{\delta_s}{2} \le \Phi\left(x^{s+1}, y^{s+1}, v\right) \quad \forall v \in V.$$

When passing to the limit likewise above, we obtain inequality (11). □

Definition 1 The triple $(\hat{x}, \hat{y}, \hat{v})$ satisfying inequalities (10) and (11) shall henceforth be called a critical point of problem $(\mathcal{DC}(\mu))$. If the inequalities (10) and (11) are satisfied with certain accuracy for some point, we call this point *approximately critical*.

It can be easily seen that a critical point turns out to be a (partially) global solution of Problem $(\mathcal{DC}(\mu))$ with respect to uncoupled groups of variables (x, y) and v.

In the capacity of a stopping criterion of proposed method one can apply, for example, the inequality

$$\Phi_s - \overline{\Phi}_s \le \tau_1, \tag{14}$$

where $\tau_1 > 0$ is a given accuracy. This rule can be derived from chain (12). Besides, it is possible to show that after stopping according to rule (14) the point (x^s, y^s, v^s) turns out to be a (partially) global $(\tau_1 + \frac{\delta_s}{2})$-solution of Problem $(\mathcal{DC}(\mu))$ with respect to (x, y) and, at the same time, a partially global $(\tau_1 + \frac{\delta_{s-1}}{2} + \frac{\delta_s}{2})$-solution with respect to v. If $\tau_1 \le \frac{\tau}{2}$, $\delta_{s-1} \le \frac{\tau}{4}$, $\delta_s \le \frac{\tau}{4}$, then we obtain the τ-critical point (x^s, y^s, v^s) of problem $(\mathcal{DC}(\mu))$.

To implement the procedure of local search in problem ($\mathcal{DC}(\mu)$) , according to the logic from [17,24], it is possible to consider another variant of its implementation, in which auxiliary problems are solved in a different order (initially—with respect to v, and next—with respect to (x, y)). This version of the local search method shall henceforth be called the XY-procedure.

Next, as an example consider some results of testing the local search procedures for problem ($\mathcal{DC}(\mu)$) . We have determined experimentally that the value $\mu = 10$ of the penalty parameter μ turns out to be sufficient for convergence of local search to a feasible points $\left(\hat{x}, \hat{y}, \hat{v}\right)$ of Problem ($\mathcal{P}$) (i.e. the equality $h(\hat{x}, \hat{y}, \hat{v}) = 0$ holds).

For constructing test problems we apply the procedure elaborated in [25,26]. When applying this procedure, it is possible to construct test bilevel problems of various complexities and dimensions with all the known local and global solutions. Furthermore, the number of these (local and global) solutions is known.

The software implementing the V- and XY-procedures has been written in C++. The accuracy of the critical point was $\tau = 10^{-4}$. Auxiliary convex quadratic problems and linear programming problems have been solved by subroutines from the software package XPress-MP (http://www.dashoptimization.com/) estimated by the experts as a rather efficient tool for solving such problems. A computer with Intel Core 2 Duo 2.4 GHz CPU has been used.

The behaviour of the XY- and V-procedures was investigated on several problems of small dimension with respect to a choice of a starting point (see Table 1). To this end we have employed the following starting points:

$$\left(x_0^{(1)}, y_0^{(1)}, v_0^{(1)}\right) = (0, 0 \ldots, 0), \quad \left(x_0^{(2)}, y_0^{(2)}, v_0^{(2)}\right) = (3, 3 \ldots, 3),$$
$$\left(x_0^{(3)}, y_0^{(3)}, v_0^{(3)}\right) = (3, 0, 3, 0 \ldots, 3, 0).$$

The following denotations have been used in the Table 1:
$Name$ is the name of the example in format "$m + n\_k$", where m is the dimension of vector x, n is the dimension of vector y, k is the number of the example of given dimension (note, the number of constraints imposed on the upper level of the bilevel problem generated is $2m$, and the number of constraints imposed on the lower level of the generated bilevel problem is $2n$);

$\Phi_X^{(1)}, \Phi_X^{(2)}, \Phi_X^{(3)}$ are the values of the goal function of Problem ($\mathcal{DC}(\mu)$) at critical points, which have been obtained by the XY-procedure, while starting from the points $(x_0^{(1)}, y_0^{(1)}, v_0^{(1)})$, $(x_0^{(2)}, y_0^{(2)}, v_0^{(2)})$ or $(x_0^{(3)}, y_0^{(3)}, v_0^{(3)})$, respectively; $\Phi_V^{(1)}, \Phi_V^{(2)}, \Phi_V^{(3)}$ are similar values obtained by the V-procedure.

Φ^* is the value of the goal function of problem ($\mathcal{DC}(\mu)$) for the known global solution of appropriate bilevel problem.

The total run time of the algorithms has turned out to be rather small (less than 0.1 s), and this information has not been included into the table. The number of iterations of the algorithms before fulfillment of the stopping criterion was two.

It can readily be seen from Table 1 that if we use the starting points $(x_0^{(1)}, y_0^{(1)}, v_0^{(1)})$ and $(x_0^{(2)}, y_0^{(2)}, v_0^{(2)})$ then the variants of local search allow to find the global solutions for 5 of 20 test problems (the corresponding results are represented in bold face in the table). However, if one uses the starting point $(x_0^{(3)}, y_0^{(3)}, v_0^{(3)})$ then only the V-procedure of local search allows to find a global solution and only for the two test problems of dimension $2 + 2$. Therefore, this point is considered to be suitable for the purpose of verification of efficiency of the global search.

Table 1 Local search from various starting points

Name	$\Phi_X^{(1)}$	$\Phi_X^{(2)}$	$\Phi_X^{(3)}$	$\Phi_V^{(1)}$	$\Phi_V^{(2)}$	$\Phi_V^{(3)}$	Φ^*
2 + 2_1	−2.00	2.00	6.00	2.00	−2.00	**−6.00**	**−6.00**
2 + 2_2	**−2.00**	14.00	6.00	14.00	**−2.00**	6.00	**−2.00**
2 + 2_3	−2.00	22.00	25.00	22.00	−2.00	**-6.00**	**−6.00**
2 + 2_4	**−2.00**	34.00	25.00	34.00	**−2.00**	6.00	**−2.00**
2 + 2_5	**−2.00**	54.00	25.00	54.00	**−2.00**	25.00	**−2.00**
4 + 4_1	−4.00	−8.00	0.00	−8.00	−4.00	−12.00	−16.00
4 + 4_2	−4.00	4.00	0.00	4.00	−4.00	0.00	−12.00
4 + 4_3	−4.00	24.00	19.00	24.00	−4.00	0.00	−12.00
4 + 4_4	−4.00	36.00	31.00	36.00	−4.00	0.00	−8.00
4 + 4_5	−4.00	56.00	31.00	56.00	−4.00	19.00	−8.00
6 + 6_1	−6.00	**−30.00**	−18.00	**−30.00**	−6.00	−18.00	**−30.00**
6 + 6_2	−6.00	6.00	6.00	6.00	−6.00	−6.00	−18.00
6 + 6_3	**−6.00**	42.00	18.00	42.00	**−6.00**	18.00	**−6.00**
6 + 6_4	−6.00	58.00	25.00	58.00	−6.00	25.00	−14.00
6 + 6_5	−6.00	66.00	44.00	66.00	−6.00	13.00	−18.00
10 + 10_1	−10.00	−6.00	1.00	−6.00	−10.00	−18.00	−42.00
10 + 10_2	−10.00	26.00	13.00	26.00	−10.00	1.00	−38.00
10 + 10_3	−10.00	70.00	44.00	70.00	−10.00	13.00	−30.00
10 + 10_4	−10.00	114.00	56.00	114.00	−10.00	44.00	−22.00
10 + 10_5	−10.00	138.00	68.00	138.00	−10.00	56.00	−14.00

4 Global search

As shown above, local search procedures have not provided for obtaining a global solution even for small-dimensional problems because it is possible to guarantee only obtaining a critical point on this stage. In this connection, we propose some procedures of Global Search for such problems as ($\mathcal{DCC}$) and ($\mathcal{DC}(\mu)$), which allowing to escape from a critical point obtained. These are based on Global Optimality Conditions [16] and on the corresponding Global Search Strategies for the d.c. programming problems with d.c. constraints and with a d.c. goal function [19,20].

Remember that Global Optimality Conditions (GOC) for d.c. minimization problem

$$\phi(x) = g_0(x) - f_0(x) \downarrow \min_x, \quad x \in D, \qquad (\mathcal{DC}_0)$$

where $x \in I\!R^n$; g_0, f_0 and D are convex, are as follows. If $z \in Sol\ (\mathcal{DC}_0)$ then

$$\forall (y, \beta) \in I\!R^n \times I\!R: \quad f_0(y) = \beta - \zeta, \quad \zeta := \phi(z), \qquad (A)$$

$$g_0(y) \le \beta \le \sup(g_0, D), \qquad (B)$$

$$g_0(x) - \beta \ge \langle \nabla f_0(y), x - y \rangle \quad \forall x \in D. \qquad (C)$$

Furthermore, these GOC possess so called algorithmic (constructive) property. It means that if GOC are broken down one can construct a feasible point which is better than the point z under scrutiny.

Actually, if one was successful to find a vector $\hat{y} \in I\!R^n$, a number β verifying conditions (A) and (B), and besides a feasible vector $\hat{x} \in D$, such that inequality (C) is violated

$$g_0(\hat{x}) < \beta + \langle \nabla f_0(\hat{y}), \hat{x} - \hat{y} \rangle,$$

then due to convexity of function $f_0(\cdot)$ one has

$$\phi(\hat{x}) \triangleq g_0(\hat{x}) - f_0(\hat{x}) < f_0(\hat{y}) + \zeta - f_0(\hat{y}) = \phi(z).$$

It means that $\phi(\hat{x}) < \phi(z)$, so that $\hat{x}$ is better than z. Therefore, varying the "perturbation parameters" $(y, \beta) \in I\!R^n \times I\!R$ by solving the linearized problems

$$g_0(x) - \langle \nabla f_0(y), x \rangle \downarrow \min_x, \quad x \in D, \qquad (PL(y))$$

(where y is not obligatory feasible!) one obtains a family of starting points $x(y, \beta)$ for Local Search Procedure. Note, that no needs to try all the pair (y, β). It is sufficient, as it has been seen, to violate inequality (C) only for one pair $(\hat{y}, \beta)$. After this one moves to a new iterate $z^{k+1} := \hat{x},\ \zeta_{k+1} := \phi(z^{k+1})$, and one repeats the procedure onto the "new level" ζ_{k+1}.

On account of particularities of the scrutinized problem and the d.c. decomposition (6) of the goal function, **the global search procedure for the problem**($\mathcal{DCC}$) may be represented as follows.

Let there be given an approximate critical point (x^k, y^k) in problem ($\mathcal{DCC}$). Note, according to Proposition 3, it is sufficient only to choose a point feasible in problem ($\mathcal{P}$). Hence the following chain of operations follows.

1) Choose a number $\beta \in [\beta_-, \beta_+]$, where $\beta_- \triangleq \inf(g, D),\ \beta_+ \triangleq \sup(g, D)$ and construct an approximation

$$\mathcal{A}_k = \left\{ (u^1, z^1), \ldots, (u^N, z^N) \mid f(u^i, z^i) = \beta, \ \ i = 1, \ldots, N \right\}$$

of the level surface $U(\beta) = \{(x, v) \mid f(x, v) = \beta\}$ of function $f(x, v)$ convex with respect to (x, v). It is possible to choose an initial β_0 equal, for example, to $f(x^k, v^k)$ ($\beta_0 := f(x^k, v^k)$).
2) Beginning from (u^i, z^i), compute $\rho_k = F(x^k, y^k)$ and find a solution (x^i, y^i, v^i) of the Linearized Problem $(\mathcal{PL}_s(\rho_k))$-(7) for each $i = 1, \ldots, N$ (where $(x^0, v^0) := (u^i, z^i)$).
3) For each $i = 1, \ldots, N$ calculate new feasible (in problem ($\mathcal{P}$) points $(\hat{x}^i, \hat{y}^i, \hat{v}^i)$ generated by (x^i, y^i, v^i) and after that choose from the set of $(\hat{x}^i, \hat{y}^i, \hat{v}^i)$ the pair $(\hat{x}, \hat{y})$ best with respect to the goal function F.
4) If the value of the goal function at the point $F(\hat{x}, \hat{y})$ turns out to be better than in a current point $F(x^k, y^k)$, renewal of the latter takes place, i.e. $(x^{k+1}, y^{k+1}) := (\hat{x}, \hat{y})$, and the process is repeated (go to Step 1).

The procedure of Global Search for ($\mathcal{DC}(\mu)$) is based on the corresponding strategy of global search in problems of d.c. minimization [16] because the goal function of ($\mathcal{DC}(\mu)$) in problems of such kind may be represented as a difference between two convex functions (see also [17,24]). In combination with directed selection, in the process of increasing the the value of parameter $\mu > 0$, the following procedure of global search in Problem ($\mathcal{DC}(\mu)$) forms a method for solving such problems as ($\mathcal{BP}$) (see Proposition 1). To begin with, one needs a d.c. representation of the goal function of problem ($\mathcal{DC}(\mu)$). One can do it, for instance, as follows:

$$\Phi(x, y, v) = G(x, y, v) - H(x, y, v), \tag{15}$$

where $G(x, y, v) = \frac{1}{2}\langle x, Cx\rangle + \langle c, x\rangle + \frac{1}{2}\langle y, C_1 y\rangle + \langle c_1, y\rangle + \mu(\langle v, b\rangle + \frac{1}{4}\|v - A_1 x\|^2)$, $H(x, y, v) = \mu(\frac{1}{4}\|v + A_1 x\|^2 - \langle y, d\rangle)$ obviously are convex functions.

Now, let us begin the description of the Global Search procedure with the use of the d.c. decomposition (15). Let there be a known critical point (x^k, y^k, v^k) obtained either by the V-procedure or by the XY-procedure. Denote $\zeta_k \triangleq \Phi(x^k, y^k, v^k)$. Then the following chain of operations is fulfilled.

1) A number $\gamma \in [\gamma_-, \gamma_+]$, where $\gamma_- \triangleq \inf(G, D)$, $\gamma_+ \triangleq \sup(G, D)$ is chosen and an approximation

$$\mathcal{A}_k = \left\{\left(u^1, w^1, z^1\right), \ldots, \left(u^N, w^N, z^N\right) \mid H\left(u^i, w^i, z^i\right) = \zeta_k + \gamma, \quad i = 1, \ldots, N\right\}$$

of the level surface

$$U(\zeta_k) = \{(x, y, v) \mid H(x, y, v) = \zeta_k + \gamma\}.$$

of the convex function $H(x, y, v)$ is constructed.

2) For each point (u^i, w^i, z^i) of $\mathcal{A}_k$ the inequality

$$G\left(u^i, w^i, z^i\right) \le \gamma, \quad i = 1, \ldots, N, \tag{16}$$

is verified according to Global Optimality Conditions for the problems of d.c. minimization [16,19]. If inequality (16) is satisfied then approximation point (u^i, w^i, z^i) shall be used on further steps, otherwise, this point is hardly ever perspective in the aspect of improving the current point (x^k, y^k, v^k) with its aid.

3) Proceeding from the approximation points (u^i, w^i, z^i) not necessarily feasible (what is very suitable in the aspect of computations), the local search allowing to obtain critical points $(\hat{x}^s, \hat{y}^s, \hat{v}^s)$, $s = 1, \ldots, N$ is executed.

4) Next, the value of the goal function at each point $(\hat{x}^s, \hat{y}^s, \hat{v}^s)$ is compared to $\Phi(x^k, y^k, v^k) = \zeta_k$. If $\Phi(\hat{x}^s, \hat{y}^s, \hat{v}^s)$ for some $s \in \{1, \ldots, N\}$ is better than ζ_k, then one sets $(x^{k+1}, y^{k+1}, v^{k+1}) := (\hat{x}^s, \hat{y}^s, \hat{v}^s)$ and turn back to stage 1.

The crucial moment of above Global Search procedures consists in constructing an approximation of the level surface of the convex function, which generates the basic nonconvexity in the problem under consideration. Particularly, in problem ($\mathcal{DC}(\mu)$) an approximation $\mathcal{A}_k = \mathcal{A}(\zeta_k)$ is constructed with the help of a given set of directions [16,17,21,23,24]:

$$Dir = \left\{\left(a^1, b^1, c^1\right), \ldots, \left(a^N, b^N, c^N\right) \mid \left(a^i, b^i, c^i\right) \in I\!R^{m+n+q}, \ i = 1, \ldots, N\right\}.$$

so that the triples (u^i, w^i, z^i) are sought in the form $\left(u^i, w^i, z^i\right) = \lambda_i \left(a^i, b^i, c^i\right), i = 1, \ldots, N$, where $\lambda_i \in I\!R$ computed, while proceeding from the condition $H\left(\lambda_i (a^i, b^i, c^i)\right) = \zeta_k + \gamma, \ i = 1, \ldots, N$. The search of λ_i turns out to be very easy and analytical for the quadratic function $H(\cdot)$. Constructing the approximation in problem ($\mathcal{DCC}$) is performed similarly.

The sets Dir can be constructed with the help of the experience obtained in course of the previous computational simulations [16–24], and we have to take account of the information related to the problem statement. For instance, in the process of solving problem ($\mathcal{DCC}$), the set

$$Dir = \left\{ \left(e^l, e^j\right), \quad l = 1, \ldots, m, \quad j = 1, \ldots, q \right\},$$

where $e^l \in I\!R^m$, $e^j \in I\!R^q$ are vectors from Euclidean basis, shows itself rather competitive.

On the other hand, for the purpose of solving problem ($\mathcal{DC}(\mu)$) the following set of directions has been selected

$$Dir = \left\{ \left((x, y) + e^i, v + e^j\right), \left((x, y) - e^i, v - e^j\right), \quad i=1, \ldots, m+n, \quad j=1, \ldots, q \right\},$$

where $e^i \in I\!R^{m+n}$, $e^j \in I\!R^q$ are Euclidean basis vectors, (x, y, v) is a current critical point.

5 Testing of the global search algorithms

Preliminary testing which has been carried out for linear-linear bilevel problems ($C \equiv 0$ and $C_1 \equiv 0$) generated with the use of the method from [25], has given evidence some advantage of the second approach (with the penalty parameter μ) in comparison with the first approach on the current stage of investigations. We have managed to experimentally find the value of the penalty parameter $\mu = 10$, which showed itself quite pertinent for finding a global solution for all the test problems (see also testing of the local search). In this case, the total number of auxiliary quadratic and linear problems, which has been solved during the process of obtaining a global solution has turned out to be smaller than in the case of solving the problem with the d.c. constraint. Note, that potentialities of improving the algorithm for solving the problem with the d.c. constraint are not exhausted yet.

In the present paper, we present some results of the computational solving of the test bilevel problems with quadratic convex functions at the upper level and linear functions at the lower level obtained with the aid of the second approach (by solving problem ($\mathcal{DC}(\mu)$). So, the latter approach may presently be considered to be more effective.

To the end of conducting the procedure of testing, some series of bilevel problems of dimension from $10 + 10$ to $150 + 150$ have been generated with the use of the approach from [25,26]. The starting points for the global search method have been chosen so that the local search procedures we would not reach a global solution. For example, the point $(x_0, y_0, v_0) = (3, 0, 3, 0 \ldots, 3, 0)$ satisfies this condition, as it has been established in Sect. 3.

Note in advance that in all the test problems generated the global search algorithm has reached a global solution with the accuracy of $\varepsilon = 10^{-4}$, that is why this information has not been included into the table.

The following denotations are used in Table 2:

m is a number of variables at the upper level of the bilevel problem (n, a number of variables at the lower level, is equal to m for the test problems);
N is a number of problems in different series;
$LocSol_{avg}$ is the average number of the local solutions which are not global in the problems of series;
$Loc_{avg}(Loc_{max})$ is the average (maximum) number of start-ups of the local search procedure in course of the global search;
$St_{avg}(St_{max})$ is the average (maximum) number of iterations of the global search algorithm;
$T_{avg}(T_{max})$ is the average (maximum) operating time of the program implementing the global search algorithm.

Table 2 Numerical solving of bilevel problems

m	N	$LocSol_{avg}$	Loc_{avg}	St_{avg}	T_{avg}	Loc_{max}	St_{max}	T_{max}
10	1,000	146.2	2012.7	1.9	22.27	4,118	7	44.52
20	1,000	131284.1	3436.6	2.1	53.80	8,671	7	2:38.23
30	100	1.34×10^8	4601.5	2.0	1:29.08	15,296	6	4:54.19
40	100	1.17×10^{11}	6485.1	2.1	2:34.73	14,660	9	6:06.92
50	100	1.20×10^{14}	9352.5	2.1	4:20.76	17,473	7	8:53.17
75	10	3.78×10^{21}	8050.3	3.0	4:52.20	13,914	4	8:21.31
100	10	1.27×10^{29}	12263.8	2.8	10:25.81	20,199	4	16:59.16
125	10	4.27×10^{36}	17704.3	2.7	20:59.14	32,049	3	38:33.78
150	10	3.93×10^{44}	72245.6	17.9	2:09:39.95	286,267	80	8:06:46.38

Note, that the number of different critical points, by which Global Search Algorithm passed, and which improve the goal function of the bilevel problem of dimension $150 + 150$, reaches a rather large number $St_{max} = 80$. However, the overall number of start-ups of the local search procedure (72245.6) in such problems turnes out to be incomparable with the huge number of local solutions (3.93×10^{44}), which are not global [25,26].

We have to emphasize again that as a result of our computational simulation it is possible to conclude that we can to solve all the test problems of dimension up to 150 + 150 within some acceptable time.

6 Conclusion

In the present paper, new procedures of finding optimistic solutions in quadratic-linear and linear-linear bilevel problems have been elaborated. On the one hand, these procedures are based on the well known idea to replace the extremum constraint in the bilevel problem with KKT-conditions. On the other hand, for the purpose of solving the obtained nonconvex single-level problems, novel Global Search Algorithms based on the Global Search Theory from [16–24] for the d.c. programming problems have been applied. Besides, new local search algorithms for these nonconvex problems have been elaborated and tested.

The programs implementing the local and global search algorithms have been developed. The computational testing has proved the considerable efficiency of the proposed approach for a number of series of randomly generated bilevel problems up to the dimension of 150 + 150. Similar results of numerical solving of these classes of bilevel problems up to such a dimension cannot be found in the literature.

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